The Wave Equation

HOME: [The Physics of Bruce Harvey]

The first clue that light had a finite speed came from astronomical observations of the eclipses of Jupiter's moons, the timing of which were found to vary by up to 20 minutes due to variations in the distance between the earth and Jupiter. The development of the theory of electricty and magnetism had resulted in two different ways of defining the units, one based on the force between charges and other on the force between currents. The electromagnetic unit of charge in cgs units is about 3×10^{10} times the size of the electrostatic unit. When the measurement of the velocity of light matched this ratio, it pointed very clearly to a link between electromagnetism and light. Maxwell used dimensional analysis to show that the ratio was a velocity and then set about proving that light, like sound was a kind of wave motion.

Now the mathematical analysis of wave motion relies on forming a second order differential equation and solving it. Maxwell's first problem was to get some differential equations from which to derive the wave equation. He knew the laws of electricity and magnetism, so he simply expressed them in differential form. Thanks to the notation of vector field theory, we can write them quite compactly. Maxwell had to express them in co-ordinate form.

Faraday's Law	$Curl\ E = -\dfrac{\partial \vec{B}}{\partial t}$	Ampere's Law	$Curl\ \vec{H} = J + \dfrac{\partial \vec{D}}{\partial t}$
Gauss' Law	$Div\ \vec{D} = \rho$	$Div\ \vec{B} = 0$	
Definitions	$\vec{D} = \varepsilon\, \varepsilon_0\, \vec{E}$	$\vec{B} = \mu\, \mu_0\, \vec{H}$	

The key element in understanding these was to extend Gauss' Law conceptually so that it not only defined the electric flux emanating from an electric charge, but also described the displacement charge which terminated electric flux (the equation is unaltered). If the flux density is to change, then according to Maxwell, displacement charge must flow so that the current density J of a current must also include a term $\frac{\partial \vec{D}}{\partial t}$ for the rate of change of the electric flux density.

These can be solved in various ways to give different wave equations of the form:

$$\nabla^2 \vec{H} = \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2}; \qquad \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}; \qquad \ldots$$

In fact, the wave equation can be expressed in terms any one of the six descriptors $\vec{D}$, $\vec{E}$, $\vec{B}$, $\vec{H}$, ϕ, $\vec{A}$ of electric and magnetic fields simply because in electromagnetic radiation, their magnitudes are everywhere proportional to each other.

Differential equations are very difficult to solve. Students can now solve them quite easily by learning the solutions which others discovered through great effort. It is normal to quote the wave equation for plain waves moving in the x direction so that it is expressed as a function of position and time:

$$H = A\cos(x - v\,t) + B\sin(x - v\,t)$$

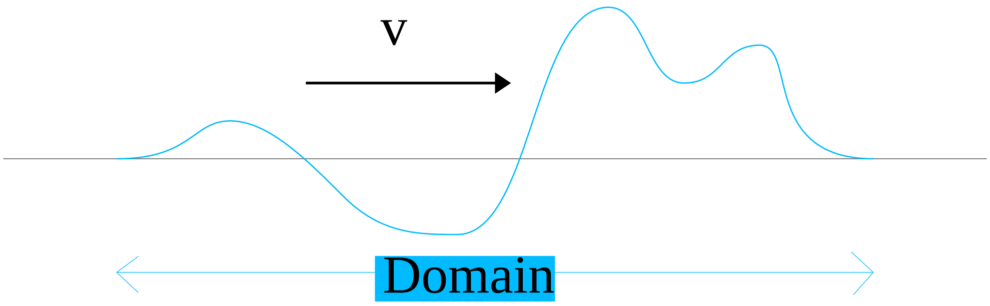
Maxwell believed that the electric and magnetic fields both "had their seat" in the "æther" and while these terms were never really understood even before we abandoned the concept of the æther, it is important to understand that Maxwell understood the electric and magnetic flux to be stationary, but changing in flux density as the wave passed.

The great success of Maxwell's equations was that they explained a whole host of phenomena observed in optics. Simply by measuring the relative permittivity ε and relative permeability μ of a sample, it was in many cases possible to calculate its refractive index and the velocity of light through it (though in some substances they are frequency dependent). More than that, it was possible to calculate the percentage of light which would be reflected back when light passed form one medium to another. Total internal reflection could be accounted for not to mention the reflection of light by conducting surfaces.

A more general approach to the wave equation leads a much better understanding of the physics. For plane waves moving with velocity v in the x direction:

$$\frac{d^2}{dx^2} f\left(x - v\,t\right) = \frac{1}{v^2}\frac{d^2}{dt^2} f\left(x - v\,t\right)$$

The function $f()$ can be more or less any variable which describes the wave. The solutions normally quoted involve sine and cosine functions or exponential functions. In fact, the solution is any single valued function which is twice differentiable and has smooth tails. Here is such a function: it is smooth, single valued, has a finite domain which moves with a velocity v to the right and its ends merge smoothly into the axis.



To understand this better, one should observe waves in water. Drop a stone in a still pond and watch the ripples. Some distance from where the stone was dropped, we can observe a wave train of ripples pass by. The water surface was flat before it arrived and is flat after it has passed. At first glance, the ripples appear to have the shape of a sine wave. Closer examination shows that the real shape is less perfect. Its peaks are not the same height and its ends are smooth. Traditional solutions use Fourier analysis to describe the shape as a series of sine and cosine functions but this involves a fiction of waves travelling in from infinite distance from both directions and happening to arrive at just the right time to form the wave train of our spreading ripples.

The wave equation uses differentiation. An important property of differentiation is that when two things add together, it does not make any difference whether we do the differentiation before or after the addition. This results in a property we call superposition. In practice it means that so long as radio stations transmit on different frequencies, we can tune a radio to receive the station of our choice. It means a single optical fibre can carry many telephone calls each modulated onto its own frequency. It means that if two stones are dropped in the pond, their ripples will pass through each other.

There is another property of waves which we observe in nature. As waves from a violent storm travel out across an ocean, they sort themselves out into nice well ordered wave trains beloved of surfers. Each wave train is several hundred meters wide and contains seven significant peaks which are progressively moving through the wave train.