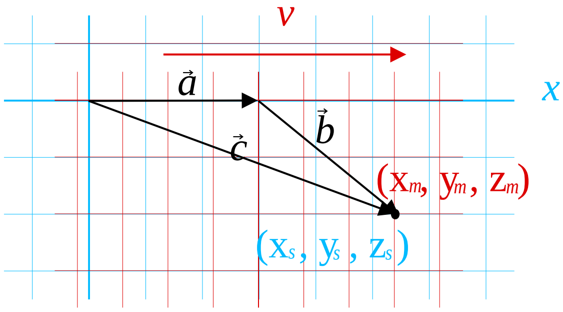
Transform from stationary to moving system

HOME: The Physics of Bruce Harvey

In relativity, we speak of an event as taking place at a point and at a time. Its descriptor consists of a point and a time recorded in the stationary system as $(x_s,\ y_s,\ z_s)\,@\,t_s$ or in the moving system as $(x_m,\ y_m,\ z_m)\,@\,t_m$. The diagram shows the red system moving at a velocity v through the blue stationary system. The x axes are coincident as are the xy planes. This view could have been recorded by a distant camera on the z axis of the stationary system. If we take the event of the light pulse, which was emitted from the origins when they were coincident, arriving at $(x_s,\ y_s,\ z_s)$ in the stationary system and $(x_m,\ y_m,\ z_m)$ in the moving system.



In the stationary system, the light pulse travels along the vector $\vec{c}$. During this time, the origin of the moving system is seen to travel along the vector $\vec{a}$. In the moving system, the light pulse is seen to travel along the vector $\vec{b}$. Clearly $\vec{a} + \vec{b} = \vec{c}$. This is a vector equation which is true in any co-ordinate system or in any units.

The vector $\vec{b} = \begin{pmatrix} x_m \\ y_m \\ z_m \end{pmatrix}$ moving system units. Vectors $\vec{a} = \begin{pmatrix} v\,t_s \\ 0 \\ 0 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix}$, both in stationary system units.
We must convert $\vec{b}$ to stationary system units by dividing its x element by γ before substituting these values in the vector equation:

$$\begin{pmatrix} v\,t_s \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \dfrac{x_m}{\gamma} \\ y_m \\ z_m \end{pmatrix} = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix}$$

From this: $x_m = \gamma\left(x_s - v\,t_s\right)$

Within the moving system, the clock at $(x_m,\ y_m,\ z_m)$ is slow compared to the clock at the origin of the moving system by $\frac{v\,x}{c^2}$ moving clock units ($x_m > 0$), so the moving master clock reads $t_m + \frac{v\,x_m}{c^2}$ which we must convert into stationary system units giving:

$$t_s = \gamma\left(t_m + \frac{v\,x_m}{c^2}\right)$$

We can substitute the above result for x_m and solve for t_m:

$$t_s = \gamma\left(t_m + \frac{v\,\gamma\,(x_s - v\,t_s)}{c^2}\right) \rightarrow t_s = \gamma\,t_m + \frac{\gamma^2\,v\,x_s}{c^2} - \frac{\gamma^2\,v\,t_s}{c^2}$$

$$t_s\left(1 + \frac{\gamma^2 v^2}{c^2}\right) = \gamma\, t_m + \frac{\gamma^2\, v\, x_s}{c^2}$$

But $1 + \frac{\gamma^2 v^2}{c^2} = \gamma^2$ therefore $\gamma^2 t_s = \gamma\, t_m + \frac{\gamma^2\, v\, x_s}{c^2}$

$$t_m = \frac{t_s - \frac{v\, x_s}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Taking this and equating components of the vector equation gives us the Lorentz transform:

$$x_m = \frac{x_s - v\, t_s}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad y_m = y_s \qquad z_m = z_s \qquad t_m = \frac{t_s - \frac{v\, x_s}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This derivation describes the physical effects caused by motion though the background in which the electromagnetic interactions take place. We have proved that it works transforming events from the stationary system to the moving system. Lorentz published a derivation of these transforms in 1903 (Proc Amst. Acad)[1vi] based on the invariance of $(c\, t)^2 = x^2 + y^2 + z^2$ and using hyperbolic functions to get the solution.

False symmetry of transform

This is a good point to discuss one of the niggles I have with the Lorentz transforms. The equation for time gives a false impression that time behaves in the same way as length. Particularly in units where $c = 1$ the two equations $x_m = \gamma\left(x_s - v\, t_s\right)$ and $t_m = \gamma\left(t_s - v\, x_s\right)$ have a high degree of symmetry and swapping x and t changes one into the other. Putting $t_s = 0$ in the first and $x_s = 0$ in the second yields $x_m = \gamma\, x_s$ and $t_m = \gamma\, t_s$. This is most deceptive because while $x_m = \gamma\, x_s$ supports the concept that a moving ruler is contracted in the direction of motion, the second would seem to indicate that a moving clock records a longer time interval. In fact, the reverse is true: a moving clock runs slow and records a shorter time interval.

This causes thinking people a lot of trouble because thinking minds notice logical inconsistencies and work away at trying to understand them. Learning minds just lap up anything they are given and ignore inconsistencies. In fact thinking minds do vast amounts of thinking at the subconscious level popping the results into the conscious mind in eureka moments. Faced with the apparent logical inconsistencies of relativity, the eureka moment never occurs and the conscious mind remains sceptical.

To understand this apparent inconsistency, we need to look back in our derivation to the line:

$$t_s\left(1 + \frac{\gamma^2 v^2}{c^2}\right) = \gamma\, t_m + \frac{\gamma^2\, v\, x_s}{c^2}$$

where the first terms of each side are t_s and $\gamma\, t_m$ and have the correct relationship to each other. The problem is that the effect of synchronisation errors is far greater. So when we take into account the two terms representing the synchronisation error, the left hand side becomes $t_s\left(1 + \frac{\gamma^2 v^2}{c^2}\right)$ which due to the algebra of γ simplifies to $\gamma^2\, t_s$ so that we now compare $\gamma^2\, t_s$ to $\gamma\, t_m$ reversing the relationship.

The reader should ponder this matter for a while: it has taken the author a mere 41 years to come to this eureka moment.