Combining Lorentz transforms

HOME: [The Physics of Bruce Harvey]

If we have three observers moving relative to one another and they choose origins such that all three are coincident at some moment, then it should be possible for them to set up co-ordinate grids and populate them with clocks according to the rules.

In what might be called a Lorentz-Poincaré universe, we can identify one of the observers S as being at rest relative to the æther. His clocks are all in absolute synchronisation and he can judge a moving line to be parallel to a fixed line because the two lines are coincident at some moment in time. The other two observers A and B will not have their local clocks absolutely synchronised and will see two lines which S judges parallel to be at an angle to each other with the point of intersection moving along the lines as one apparently moves past the other.

The educated reader will have been taught to believe that we live in an Einstein universe where all three systems have equal status and such concepts as stationary and moving are only relative terms. This universe has the property that the laws of physics are the same for all observers leading to the inference that the physical effects of contraction in length, increase in mass and slowing of clocks are not physical effects at all, but are merely artefacts of observation caused by the relative motion of the observers and observed.

If one admits the validity of the Lorentz transforms, then the proof given here will also work in an Einstein universe, but the author disputes the validity of the Lorentz transforms in an Einstein universe. Since in an Einstein universe, no system suffers a real contraction in length, their grid lines beat out perfectly synchronised Newtonian time as they pass one another. From this clocks can be perfectly calibrated and synchronised and the whole basis of Einstein's derivation collapses.

In either case, three observers can set up a total of six co-ordinate grids populated with clocks. We shall refer to a co-ordinate grid populated with synchronised clocks a co-ordinate system. We introduce the notion:

Ab is a co-ordinate system set up by Observer A aligned with observer B

Ab_P is an event at point P described in terms of Ab

Cd_Ab is the transform matrix to change the co-ordinates and time of an event from Ab to Cd

Using this notation, the rules of matrix multiplication allow a cancelling rule.

$$Cd\_Ab \cdot Ab\_P = Cd\_P$$

We can write equations showing both cancelling and expansion using this notation for both a Lorentz transform of an event and the multiplication of one transform matrix by another.

	Transform of point	Multiplication of transforms
Cancelling	$As\_Ab \cdot Ab\_P = As\_P$	$As\_Ab \cdot Ab\_Ba = As\_Ba$
Expansion	$As\_P = As\_Ab \cdot Ab\_P$	$As\_Ba = As\_Ab \cdot Ab\_Ba$

The multiplication of one transform matrix by another is best understood when we apply the transforms to an event.

$$As\_Ab \cdot Ab\_Ba \cdot Ba\_P = As\_Ba \cdot Ba\_P$$

Which is to say that transforming the event P from Ba to Ab and then from Ab to As is the same as transforming it from Ba to As.

Our three observers, S, A and B set up a total of six co-ordinate systems: Sa, As, Sb, Bs, Ab and Ba. There are a total of 36 possible transformations between them: *Sa_Sa*, *Sa_As*, *Sa_Sb*....... *Ba_Ba* which we can divide into various forms:

Six matrices of the form *Ab_As* are rotations.

Six matrices of the form *Ab_Ab* are equal to the identity element I.

The two matrices *Sa_As* and *Sb_Bs* are known to be Lorentz transforms

Another four matrices of the form *Ab_Ba* might be Lorentz transforms.

The remaining transforms are said to be Lorentz invariant. However, they are not proper Lorentz transforms because they do not satisfy rules ii, iii and iv for setting up co-ordinate systems for a standard Lorentz transform, though they do preserve the geometry. Expressed as matrices, they lack the symmetry of Lorentz transform matrices.

The properties of a group are satisfied if we have:

(i)A set

(ii)An operation defined to combine any two members of the set to get another member of the set

(iii)The operation is Associative

(iv)There is an identity element I

(v)Every element has an inverse with which it combines to give I

We must make two points here. The first is that a group is a mathematical entity and its properties are independent of the nature of its elements. The second is that there are a number of ways of defining a group. Mathematical entities often have more properties than are needed to define them and can be defined by different subsets of their properties.

One of the properties of a group is *closure* as implied in (ii). The counting numbers 0, 1, 2, 3.... under addition form a group, but it is an infinite group because if we learn how to count up to a thousand, we might want to add 1000 + 1000 which requires us to extend our number system to count up to 2000. On the other hand, we can get quite small sets which form groups under particular operations. For instance, in the arithmetic of complex numbers, the cube roots of 1 under multiplication form a group with only three members.

Obviously, our set of 36 meaningful transforms and the operation of combining two of them cannot form a group because there is a restriction on combining them. Only operations of the form *As_Ab* · *Ab_Ba* which allow cancellation can be allowed. Therefore there is no closure and the 36 transforms do not form a group.

Our quest is to discover whether or not the two transforms Ab_Ba and Ba_Ab are Lorentz transforms.

The problem has to be set up carefully. Consider three observers S, A and B who can each identify an origin such that the three origins are coincident at some moment and together set up the six co-ordinate systems Sa, As; Ab, Ba; Bs and Sb according to the rules for Lorentz transforms. So that we can draw meaningful diagrams we allocate arbitrary conditions that:

S observes: speed of A u = 3/5, speed of B v = 4/5 and the angle between their paths to be $\theta = \frac{\pi}{6}$ or 30°.

We ask them to define the positive direction of x to be from S to A; from B to A and from S to B. We ask them to orientate their y axes in the plane of the triangle SAB. Each observer has two co-ordinate systems. We ask them to measure the angles through which they would rotate one x axis to sit on the other.

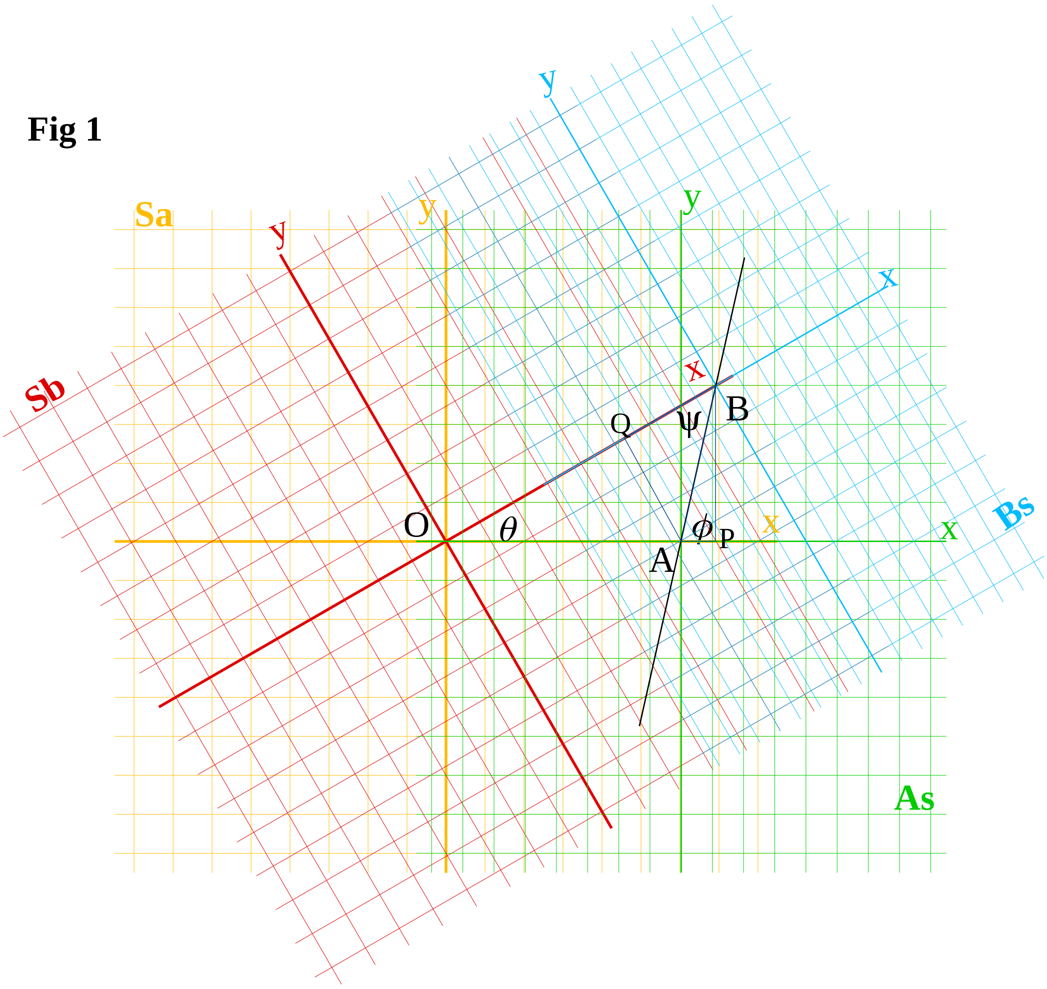
We can form the two Lorentz transforms Sa_As and Sb_Bs (which we know are valid) and form the rotation matrices which each observer uses between his two co-ordinate systems. Then we can try to form the two transforms Ab_Ba and Ba_Ab by combining transforms. Then we test to see if these transforms are indeed Lorentz transforms and if one is the inverse of the other.

Fig.1 assumes a Lorentz-Poincaré universe and is the view as S, the observer in the stationary system, sees it. Units could be any units in which the speed of light is 1. We are thinking in terms of nanoseconds and light-nanoseconds (slightly under 1 foot) and the view is taken 10 nanoseconds after the origins of the three observer's co-ordinate systems were at O. A stationary system camera at (0, 0, 1000) taking a picture at time 1010ns would record this view.

Only four of the six co-ordinate systems are represented by grids. Sa in yellow and Sb in red are uncontracted. The grid of As in green is contracted by a factor of 4/5 due to its observed speed of 3/5c and that of Bs in blue by a factor of 3/5 due to B's observed velocity of 4/5c. The xy planes of all co-ordinates grids lie in the plane of the triangle OAB.

The clocks of A and B are are affected by Lorentz contraction and mass increase causing them to run slow, so at the time of 10 nanoseconds (By S's master clock) the master clocks of A and B read 8 and 6 nanoseconds respectively. All of S's clocks are in perfect synchronisation and (in the absence of gravity) keep perfect time.

Fig 1



We can write the matrix transforms for As_Sa, Bs_Sb and Sb_Sa:

$$As\_Sa = \begin{pmatrix} \gamma_a & 0 & 0 & -\gamma_a u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_a u & 0 & 0 & \gamma_a \end{pmatrix} \quad Bs\_Sb = \begin{pmatrix} \gamma_b & 0 & 0 & -\gamma_b v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_b v & 0 & 0 & \gamma_b \end{pmatrix} \quad Sb\_Sa = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

There is no doubt that we can use matrix algebra to form the transforms Ba_Ab and Ab_Ba and the they will be inverse of each other. The question is whether or not they will be Lorentz transforms. Let us expand the transform *Ba_Ab*:

$$Ba\_Ab = Ba\_Bs \cdot Bs\_Sb \cdot Sb\_Sa \cdot Sa\_As \cdot As\_Ab$$

The only problem is that we do not know how to measure the angles which have to be used in the rotations Ba_Bs and As_Ab. So we just call them ψ and ϕ, form the matrices and do the multiplication. With a recent edition of Mathcad, that should be simple.

$$\begin{pmatrix} \cos(Bs\_\psi) & \sin(Bs\_\psi) & 0 & 0 \\ -\sin(Bs\_\psi) & \cos(Bs\_\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma_b & 0 & 0 & -\gamma_b \cdot v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_b \cdot v & 0 & 0 & \gamma_b \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma_a & 0 & 0 & \gamma_a \cdot u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma_a \cdot u & 0 & 0 & \gamma_a \end{pmatrix} \cdot \begin{pmatrix} \cos(As\_\phi) & -\sin(As\_\phi) & 0 & 0 \\ \sin(As\_\phi) & \cos(As\_\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We need three lines to show the result; column 1, column 2 and columns 3 and 4.

$$
\left(\begin{array}{c}
\cos(As\_\phi)\cdot\gamma_a\cdot\cos(Bs\_\psi)\cdot\gamma_b\cdot\cos(\theta) - \cos(As\_\phi)\cdot\gamma_a\cdot\sin(Bs\_\psi)\cdot\sin(\theta) - \cos(As\_\phi)\cdot\cos(Bs\_\psi)\cdot\gamma_b\cdot v\cdot\gamma_a\cdot u + \sin(As\_\phi)\cdot\cos(Bs\_\psi)\cdot\gamma_b\cdot\sin(\theta) + \sin(As\_\phi)\cdot\sin(Bs\_\psi)\cdot\cos(\theta) \\
-\cos(As\_\phi)\cdot\gamma_a\cdot\sin(Bs\_\psi)\cdot\gamma_b\cdot\cos(\theta) - \cos(As\_\phi)\cdot\gamma_a\cdot\cos(Bs\_\psi)\cdot\sin(\theta) + \cos(As\_\phi)\cdot\sin(Bs\_\psi)\cdot\gamma_b\cdot v\cdot\gamma_a\cdot u - \sin(As\_\phi)\cdot\sin(Bs\_\psi)\cdot\gamma_b\cdot\sin(\theta) + \sin(As\_\phi)\cdot\cos(Bs\_\psi)\cdot\cos(\theta) \\
0 \\
-\cos(As\_\phi)\cdot\gamma_b\cdot v\cdot\cos(\theta)\cdot\gamma_a + \cos(As\_\phi)\cdot\gamma_b\cdot\gamma_a\cdot u - \gamma_b\cdot v\cdot\sin(\theta)\cdot\sin(As\_\phi)
\end{array}\right.
$$

$$
\begin{array}{c}
-\sin(As\_\phi)\cdot\gamma_a\cdot\cos(Bs\_\psi)\cdot\gamma_b\cdot\cos(\theta) + \sin(As\_\phi)\cdot\gamma_a\cdot\sin(Bs\_\psi)\cdot\sin(\theta) + \sin(As\_\phi)\cdot\cos(Bs\_\psi)\cdot\gamma_b\cdot v\cdot\gamma_a\cdot u + \cos(As\_\phi)\cdot\cos(Bs\_\psi)\cdot\gamma_b\cdot\sin(\theta) + \cos(As\_\phi)\cdot\sin(Bs\_\psi)\cdot\cos(\theta) \\
\sin(As\_\phi)\cdot\gamma_a\cdot\sin(Bs\_\psi)\cdot\gamma_b\cdot\cos(\theta) + \sin(As\_\phi)\cdot\gamma_a\cdot\cos(Bs\_\psi)\cdot\sin(\theta) - \sin(As\_\phi)\cdot\sin(Bs\_\psi)\cdot\gamma_b\cdot v\cdot\gamma_a\cdot u - \cos(As\_\phi)\cdot\sin(Bs\_\psi)\cdot\gamma_b\cdot\sin(\theta) + \cos(As\_\phi)\cdot\cos(Bs\_\psi)\cdot\cos(\theta) \\
0 \\
\sin(As\_\phi)\cdot\gamma_b\cdot v\cdot\cos(\theta)\cdot\gamma_a - \sin(As\_\phi)\cdot\gamma_b\cdot\gamma_a\cdot u - \gamma_b\cdot v\cdot\sin(\theta)\cdot\cos(As\_\phi)
\end{array}
$$

$$
\left.\begin{array}{cc}
0 & \gamma_a\cdot u\cdot\cos(Bs\_\psi)\cdot\gamma_b\cdot\cos(\theta) - \gamma_a\cdot u\cdot\sin(Bs\_\psi)\cdot\sin(\theta) - \cos(Bs\_\psi)\cdot\gamma_b\cdot v\cdot\gamma_a \\
0 & -\gamma_a\cdot u\cdot\sin(Bs\_\psi)\cdot\gamma_b\cdot\cos(\theta) - \gamma_a\cdot u\cdot\cos(Bs\_\psi)\cdot\sin(\theta) + \sin(Bs\_\psi)\cdot\gamma_b\cdot v\cdot\gamma_a \\
1 & 0 \\
0 & -\gamma_b\cdot v\cdot\cos(\theta)\cdot\gamma_a\cdot u + \gamma_b\cdot\gamma_a
\end{array}\right)
$$

All we have to do is to see if we can equate this to a standard Lorentz transform matrix and solve for ψ and ϕ.

[There are two problems: fear and lack of computing capacity. As stated above, the author's early attempts to solve this problem failed due to lack of computing capacity and software functionality. Returning to the problem in March 2005, the author eventually cracked the problem.]

If this is a Lorentz transform, then the opposite corner elements should be equal and the rest equate to 0 or 1. Equating elements row 2 col 4 and row 3 col 3 to zero gives:

$$tan\,\phi \;=\; \frac{v\,sin\,\theta}{\gamma_a\left(v\,cos\,\theta\;-\;u\right)} \qquad tan\,\psi \;=\; \frac{u\,sin\,\theta}{\gamma_b\left(u\,cos\,\theta\;-\;v\right)}$$

Dividing row 4 col 1 by row 4 col 4 gives the speed of A and B which they measure relative to each other

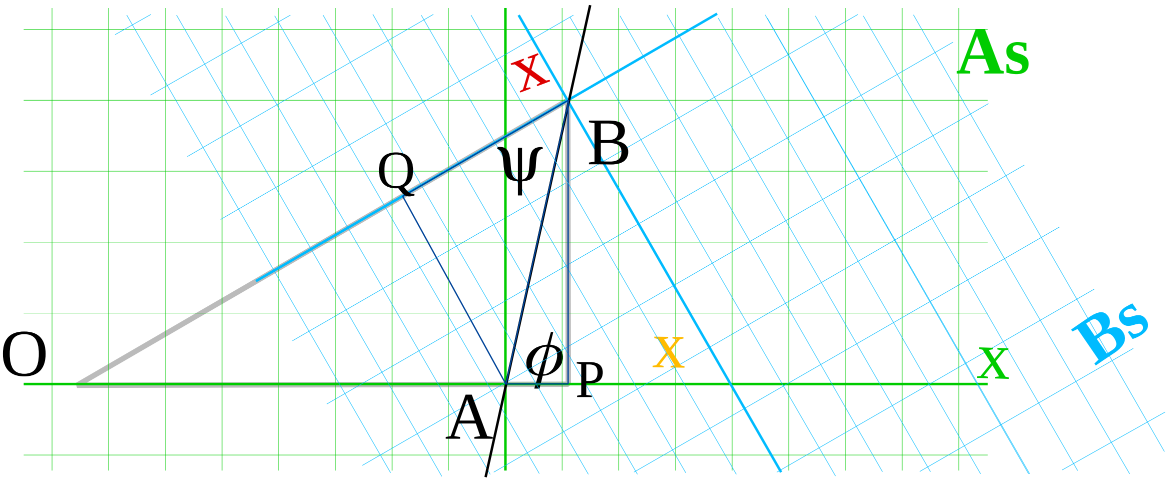
$$w \;=\; \frac{\sqrt{(1\;-\;u\,v\,cos\,\theta)^2\;-\;(1\;-\;u^2)\,(1\;-\;v^2)}}{1\;-\;u\,v\,cos\,\theta}$$

It is not possible to use Mathcad to substitute these values back into the matrix and reduce it to a Lorentz transform matrix because the algebra is too complex. It is however a simple matter to obtain numerical solutions to particular examples and substitute them. This has been done enough times to ensure the solution is correct. For the example given in the diagram:

$$\psi \;=\; 32.70° \qquad \phi \;=\; 73.82° \qquad w \;=\; 0.5702\,c \qquad \gamma \;=\; 1.217$$

In hindsight, we can see that these angles could have been obtained from fig.1. They are not the angles which we would measure from fig 1 with a protractor, but are the angles A and B would measure with their contracted protractors. The contracted grids preserve the tangents of the angles.

Fig 2



The enlarged part of the diagram (Fig.2) shows two right angled triangles APB and BQA with the blue and green grids. From this, it is easy to derive the equations:

$$\tan\phi = \frac{PB}{AP} = \frac{v\,t\sin\theta}{\gamma_a(v\,t\cos\theta - u\,t)} \qquad \tan\psi = \frac{QA}{BQ} = \frac{u\,t\sin\theta}{\gamma_b(u\,t\cos\theta - v\,t)}$$

The constants $\gamma_a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\gamma_b = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ respectively turning stationary system units into green and blue grid units.

The angles θ, $(\pi - \phi)$ and ψ of the triangle OAB as measured by S, A and B do not add to 180° because A and B measure their angles with protractors contracted in the direction of their motion through the stationary system.

Thus we have proved that:

The validity of the Lorentz transforms from the stationary system to a moving system guarantees that they are valid between any two moving systems.