Magnetic forces and induction

HOME: The Physics of Bruce Harvey

This section broadly follows classical EM theory, but it is more explicit and rigorous removing some of the confusion inherited from the use of older systems of units in which the distinction between $\vec{B}$ and $\vec{H}$ was less clear.

Introduction

There is only one slow moving magnetic field.

$$\vec{B} = \mu_0 \sum_{j \in C} \vec{v}_j \wedge \vec{D}_j \qquad C = \{\text{all elementary charged particles}\} \tag{1}$$

While it is convenient to speak of distinct magnetic fields such as "the earth's magnetic field" and "the field of this solenoid", in reality these fields are distinct only as a partitioning of the set of all elementary charged particles into subsets whose motion makes a specific contribution to the formation of $\vec{B}$.

$$\vec{H}_{earth} = \sum_{i \in E} \vec{v}_i \wedge \vec{D}_i \qquad E = \{\text{electrons in earth's magnetic core}\}$$

$$\vec{H}_{solenoid} = \sum_{j \in S} \vec{v}_j \wedge \vec{D}_j \qquad S = \{\text{conduction band electrons forming current in solenoid}\} \tag{2}$$

Where the velocities are measured relative to the respective bodies.

In addition to the universal "slow" magnetic field, the magnetic fields of radio waves and photons exist independently of one another passing through the universal field and each other.

It is also to be noted that the cross sectional area of the quantum strands of flux in a magnet contains tens of thousands of atoms. Therefore the magnetic flux which exists at the atomic scale must be able to coexist with the flux within the magnet. The author is still pondering this matter.

These equations involve the velocities of elementary charged particles. Velocity is a relative measure and must be interpreted with care. In its most fundamental form, equation (1) requires both conduction band electrons and their associated lattice ions to be included and the velocities to be measured relative to the background formed by the coexisting electric fields of all elementary charged particles. But we can usually restrict the summations to the motions of a definite set of electrons within an object (planet, ... wire) with their velocities measured relative to that object as in equation (2).

[If we consider the summation $\vec{H}_{solenoid} = \sum_{j \in solenoid} \vec{v}_{j,background} \wedge \vec{D}_j$ where the velocities are measured relative to the background, then we can identify for almost every electron a set of elementary charged particles comprising a lattice ion whose net charge is equal and opposite to that of the electron. Writing $\vec{v}_{j,background}$ as $\vec{v}_{atom,\, background} + \vec{v}_{j,\, atom}$ for each of these subsets, we can factorise and cancel terms involving the velocity measured relative to the background. While this still leaves those terms representing a net electrical charge on the solenoid, two factors justify our ignoring their contribution. The first is that their number is tens of orders of magnitude less than the total number of atoms. The second is the fact that all motion through the background causes the relativistic effects which make it impossible to detect our motion through the background.]

The energy density of a magnetic field is according to the classical theory $Q = \frac{1}{2}\vec{B} \cdot \vec{H}$. If we now rewrite this in terms of the motions of individual charges, the energy $\delta\mathscr{E}$ within a volume element $\delta\tau$ is:

$$\delta\mathscr{E} = \frac{1}{2}\mu_0 \sum_{j \in C} \vec{v}_j \wedge \vec{D}_j \cdot \sum_{i \in C} \vec{v}_i \wedge \vec{D}_i \,\delta\tau \tag{3}$$

This is the fundamental equation of all electromagnetic interaction. Changes in the energy content of the universal magnetic field are accomplished by an exchange of energy with elementary charged particles through the generation of forces.

- As the current through a circuit (current loop or solenoid) is increased, work must be done to increase the energy content of its magnetic field. As energy drains from the conduction band electrons, a force is generated upon each one resisting the emf of the circuit. The net effect of these forces is described as the "induced emf".

- As the current through a circuit is reduced, energy from the magnetic field must be adsorbed doing work and generating a force on each conduction band electron. The net effect of these forces is again described as the "induced emf".

- As a wire moves in relation to a magnetic field, the same action may generate an emf in it depending on whether or not the quantity of flux threading it is affected.

- When a current flows through a wire which is in a magnetic field, this same action generates a force on the wire.

While the action in the first two cases is simple to understand, the latter cases require a more subtle level of understanding. Never the less, all four physical phenomena are manifestations of the same action of nature **which exists primarily to give matter the property of inertia**. The four phenomena described above are but a by-product of its nature.

Energy of a tube of magnetic flux

The energy density of magnetic flux is:

$$Q_m = \frac{\vec{B} \cdot \vec{H}}{2}$$

Both $\vec{H}$ and $\vec{B}$ are free of divergence. This allows us to perform our integrations over the geometry of the magnetic field. We do this by dividing space into elements of volume which are tubes with sides everywhere parallel to the magnetic field. It took a century to discover that this mathematical device mimics nature's action in forming a magnetic field from quantised strands of magnetic flux. Although the calculus uses a tube containing a quantity of flux $\delta\Phi = \vec{B} \cdot \delta\vec{A}$, where δA is the area of cross section of the tube, and takes the limit as $\delta A \to 0$, the size of the quanta of magnetic flux in relation to macroscopic magnetic fields is sufficiently small to make the summation over quantum strands equal to the integral.

We consider an element of volume, length δl of a quantum strand of flux Φ_0 of cross section δA. The volume of this element is $\delta V = \delta l \, \delta A$. The flux density is:

$$\vec{B} = \frac{\Phi_0}{\delta A}$$

The energy content of an element of volume is:

$$\delta\mathscr{E} = Q_m \,\delta V = \frac{\frac{\Phi_0}{\delta A}\hat{l} \cdot \vec{H}}{2}\,\delta l\,\delta A = \frac{1}{2}\Phi_0 \vec{H} \cdot \delta l$$

The energy within the quantum strand of flux is:

$$\mathscr{E} = \frac{1}{2}\Phi_0 \int \vec{H} \cdot dl$$

This is very important because, as we have proved in our derivation of Ampere's law, this integral is equal to the current threading the loop formed by the stand of flux:

$$\int \vec{H} \cdot dl = I$$

Any tube of flux Φ will be made of quantum strands of flux.

$$\text{Therefore} \qquad \mathscr{E}_{\Phi_0} = \frac{1}{2} I\,\Phi_0, \qquad \mathscr{E} = \frac{1}{2} I\,\Phi$$

Inductance of a circuit

When a current flows through a circuit, a magnetic field results. (Though in some circuits like a pair of twisted wires the field may be very local) The magnetic field contains energy and work must be done to create it. If the current is switched off, the magnetic field will collapse and its energy content must do work to escape. These effects are proportional to a property of the circuit we call inductance.

The energy content is mainly determined by how tightly the magnetic field can wrap itself around the conductor(s). Suppose we have a loop 2m in diameter formed from wire 0.2mm thick and with a current of say 10 amps flowing through it. Changing the shape of the loop will have very little effect on the energy content of the magnetic field, but a similar loop of 0.1mm thick wire carrying the same current will produce a magnetic field with 4 times the energy content. If on the other hand, we measure the form of the magnetic field away from the surface of the wire, it depends only on the form of the loop and the current it carries. We must understand that magnetic fields have these two very different properties, but also understand that this is a great inconvenience to the engineer who seeks to design circuits in which the two properties loose their distinction. The most simple device of this nature is the coil (or solenoid).

If we tightly wind two coils say,

A) 25 turns of 0.2mm thick wire on a tube of diameter 3 cm

B) 50 turns of 0.1mm thick wire on a tube of diameter 3 cm

Identical magnetic fields will be produced by a current of 2 amps through coil A and one of 1 amp through coil B. The form of the coils determines that the shortest loop of magnetic flux is 10cm long and from the engineers point of view, the energy content of their magnetic fields is now far more predictable. The physicist needs to recognise when an equation represents the raw behaviour of nature and when it is determined by the engineer's skilled design. Faraday's classical experiment used a ring of soft iron with two coils wound on it. That was a highly engineered piece of apparatus with all the loops of magnetic flux contained in the iron core and being of nearly equal length.

If on the other hand, we wish to develop a general theory for any two circuits, we have to be careful not to carry assumptions across from the well engineered predictable situation to the more complex general situation.

We can now identify the energy content of the magnetic field as the sum of the energy content of its individual quantum flux strands. For an isolated circuit carrying a current i which generates a flux Φ:

$$\mathscr{E} = \frac{1}{2}\Phi\, i$$

Let us assume two circuits P and Q carrying currents i_p and i_q which if far from each other and any other influence would generate magnetic fields $\vec{B}_p$ and $\vec{B}_q$ with magnetic flux Φ_p and Φ_q.

Now let us bring our two circuits into each other's vicinity while maintaining the currents. What happens depends on the orientation of the circuits and the direction of the currents. There are two possible situations. Either their magnetic fields will remain separate, or else they will share some loops of flux.

If they share loops of flux, there will be an attractive force between them. We might expect that their magnetic fields will contain less energy than they did when separate because the attractive forces did work as the coils came together. But this is not the case for if we bring two similar circuits into close contact and maintain the current through each, the energy content of the magnetic field is increased by a factor of nearly four. This is because as we bring the circuits together, an emf is generated in each, opposing its current. If we use circuitry to maintain the currents, the voltage across each circuit will increase to maintain the current and work will be done against the induced emf. Energy is also being used all the time to overcome the electrical resistance of the coils. The energy calculations require care. A change in magnetic energy $E_{mag} - E_{mag0}$ over a period of time, plus the mechanical energy generated being equal to sum of the difference between energy delivered to coil and energy used overcoming its resistance for that time.

$$\left(E_{mag} - E_{mag0}\right) + E_{mech} = E_{elc_1} - E_{resit_1} + E_{elc_2} - E_{resit_2}$$

If the two circuits do not share any loops of flux, then the action of the current in each will be such as to oppose the action of the other current. As we bring the circuits together, the number of strands of flux threading each will be reduced as they are adsorbed into the surface of the conductor of each circuit. This action will induce emfs in each circuit in the direction of the current and the circuitry maintaining each current will reduce the voltage as each induced emf does part of the work in maintaining the current through the resistance of each circuit. The coils will repel each other and we must do mechanical work. The equation becomes:

$$\left(E_{mag} - E_{mag0}\right) - E_{mech} = E_{elc_1} - E_{resit_1} + E_{elc_2} - E_{resit_2}$$

and the quantity of electrical energy supplied is now less than that needed to overcome the resistances of the coils.

So much can be gained from any good textbook to be found in a university department of Electrical Engineering.

If the fields do share quantum strands of flux which thread both, as they are brought together, strands surrounding each circuit break with themselves and join with each other to form single strands. Their energy content remains constant as they do this because the two strands have energy $\frac{1}{2}\,\Phi_0\, i_p + \frac{1}{2}\,\Phi_0\, i_q$ and the new strand which they form has energy $\frac{1}{2}\,\Phi_0\left(i_p + i_q\right)$. The gain in energy of the field is explained by the fact that new stands of flux emerge from the surface of the conductor of each circuit. So as two coils which are attracting each other are brought together, there is an increase in the total flux threading each coil.

Classical physics asserts correctly that $\vec{B} = \mu_0\,\vec{H}$, but we have to be careful in our interpretation of this. When the two circuits P and Q are isolated, we can write:

$$\vec{B}_{p,0} = \mu_0\,\vec{H}_p \qquad \text{and} \qquad \vec{B}_{q,0} = \mu_0\,\vec{H}_q$$

When the two circuits are in each others proximity,we may only write:

$$\vec{B} = \mu_0\left(\vec{H}_p + \vec{H}_q\right)$$

The classical theory has an ambiguity in its interpretation because of our lax use of the term "magnetic flux" to refer to both $\vec{B}$ and $\vec{H}$. (The correct terminology is to call $\vec{B}$ "the flux of magnetic induction" and $\vec{H}$ "the flux of magnetic intensity".)

When we speak of "lines of flux cutting conductors" we are in fact referring to the lines of $\vec{H}_p = \sum_{j \in P} \vec{v}_j \wedge \vec{D}_j$ and $\vec{H}_q = \sum_{i \in Q} \vec{v}_i \wedge \vec{D}_i$. These lines are only mathematical artefacts and have different geometries from $\vec{B}$, the magnetic field which they generate together, which is:

$$\vec{B} = \mu_0 \left\{ \sum_{j \in P} \vec{v}_j \wedge \vec{D}_j + \sum_{i \in Q} \vec{v}_i \wedge \vec{D}_i \right\} = \mu_0 \left(\vec{H}_p + \vec{H}_q \right)$$

The energy density of this magnetic field is $\frac{1}{2}\vec{B} \cdot \left(\vec{H}_p + \vec{H}_q\right)$. When the coils were isolated, we could write $\vec{B} = \mu_0 \vec{H}$ showing that the real magnetic flux $\vec{B}$ has the same form as the mathematical artefact $\vec{H}$. Mathematically speaking, all of these quantities $\vec{B}$, $\vec{H}_p$, $\left(\vec{v}_i \wedge \vec{D}_i\right)$ etc. are "vector fields", but as physicists, we have to distinguish between those which are real physical entities and those which are mathematical artefacts. This subtle distinction hinges on identifying the action of nature which does the addition. Our assertion here is that each $\vec{v}_i \wedge \vec{D}_i$ acts on space, which responds to their sum, forming the magnetic field $\vec{B}$ and we can justify this from the fact that the energy content is $\frac{1}{2}\vec{B} \cdot \left(\vec{H}_p + \vec{H}_q\right)$, and not $\frac{1}{2}\left(\vec{B}_p \cdot \vec{H}_p + \vec{B}_q \cdot \vec{H}_q\right)$ or $\frac{1}{2}\mu_0 \Sigma \left(\vec{v}_i \wedge \vec{D}_i\right)^2$. The vector fields $\vec{H}_p$ and $\vec{H}_q$ are not real separate physical entities because if they were, they would each have an associated magnetic field $\vec{B}_p$ and $\vec{B}_q$ with energy densities $\frac{1}{2}\vec{B}_p \cdot \vec{H}_p$ and $\frac{1}{2}\vec{B}_q \cdot \vec{H}_q$.

We can not directly integrate $\frac{1}{2}\vec{B} \cdot \left(\vec{H}_p + \vec{H}_q\right)$ because we do not know the geometry of $\vec{B}$. However, although the vector fields $\vec{H}_p$ and $\vec{H}_q$ may be just mathematical artefacts, they have real geometry and we can use that geometry as a basis for dividing the volume integral for the energy $\mathcal{E} = \frac{1}{2} \int \vec{B} \cdot \left(\vec{H}_p + \vec{H}_q\right) d\tau$, so that we may express the integral in a simplified form using results obtained for the individual isolated circuits.

The energy stored in the magnetic field is:

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2} \int \vec{B} \cdot \left(\vec{H}_p + \vec{H}_q\right) d\tau \\
&= \frac{1}{2} \int \mu_0 \left(\vec{H}_p + \vec{H}_q\right) \cdot \left(\vec{H}_p + \vec{H}_q\right) d\tau \\
&= \frac{1}{2} \int \left(\mu_0 \vec{H}_p \cdot \vec{H}_p + \mu_0 \vec{H}_p \cdot \vec{H}_q + \mu_0 \vec{H}_q \cdot \vec{H}_p + \mu_0 \vec{H}_q \cdot \vec{H}_q\right) d\tau
\end{aligned}$$

We can readily identify the first and last terms with the energy content of the magnetic fields of the isolated circuits: $\frac{1}{2} \int \vec{B}_p \cdot \vec{H}_p \, d\tau = \frac{1}{2} \mathbf{\Phi}_{p,0} \, i_p$ and $\frac{1}{2} \int \vec{B}_q \cdot \vec{H}_q \, d\tau = \frac{1}{2} \mathbf{\Phi}_{q,0} \, i_q$.

Where $\mathbf{\Phi}_{p,0}$ and $\mathbf{\Phi}_{q,0}$ are the quantities of flux threading the isolated circuits. We can calculate them:

$$\mathbf{\Phi}_{p,0} = \int_p \vec{B}_{p,0} \cdot d\vec{A} \quad ; \quad \mathbf{\Phi}_{q,0} = \int_q \vec{B}_{q,0} \cdot d\vec{A}$$

Where the integral is over a surface bounded by the circuit.

Or in terms of the magnetic intensities, we write:

$$\mathbf{\Phi}_{p,0} = \mu_0 \int_p \vec{H}_p \cdot d\vec{A} \quad ; \quad \mathbf{\Phi}_{q,0} = \mu_0 \int_q \vec{H}_q \cdot d\vec{A}$$

Now as the circuits are brought into each other's proximity we have:

$$\mathbf{\Phi}_p = \mu_0 \int_p \left(\vec{H}_p + \vec{H}_q\right) \cdot d\vec{A} \quad ; \quad \mathbf{\Phi}_q = \mu_0 \int_q \left(\vec{H}_q + \vec{H}_p\right) \cdot d\vec{A}$$

Introducing two new quantities:

$$\Phi_{p,q} = \mu_0 \int_p \vec{H}_q \cdot d\vec{A} \quad ; \quad \Phi_{q,p} = \mu_0 \int_q \vec{H}_p \cdot d\vec{A}$$

we have:

$$\Phi_p = \Phi_{p,0} + \Phi_{p,q} \quad ; \quad \Phi_q = \Phi_{q,0} + \Phi_{q,p}$$

We may quote the classical result:

$$\Phi_{p,q} = \Phi_{q,p}$$

derived from equating the energy changes when P is brought into the vicinity of Q with those when Q is brought into the vicinity of P.

The interpretation we place on these results is as follows. When the coils are orientated such that flux linkage occurs, the number of quantum strands of flux through each circuit increases by the number of strands which link. But the quantities $\Phi_{p,q} = \Phi_{q,p}$ are defined by integrals and may be negative or positive. In the case where the orientation of the circuits is such that they repel each other, then the integrals are negative and $\Phi_{p,q} = \Phi_{q,p}$ no longer physically exists! However, the number of quantum strands of flux through each circuit is decreased by this same number.

The laws of self inductance and mutual inductance follow from our fundamental assertions